A well-balanced relaxation scheme for the Shallow Water Magnetohydrodynamic system

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The incompressible MHD system describes the evolution of a charged gas interacting with a magnetic field. In the shallow regime, the SWMHD (Shallow Water MHD) system is relevant. If the state is described with dependency on only one spatial dimension x and time t, the equations are

$$\begin{aligned}
\partial_t h + \partial_x(hu) &= 0,\\
\partial_t(hu) + \partial_x(hu^2 + P) &= -gh\partial_x z,\\
\partial_t(hv) + \partial_x(huv + P_\perp) &= 0,\\
\partial_t(ha) + u\partial_x(ha) &= 0,\\
\partial_t(hb) + \partial_x(hbu - hva) + v\partial_x(ha) &= 0,
\end{aligned}$$

with

$$P = g\frac{h^2}{2} - ha^2, \quad P_\perp = -hab,$$

where g is the gravitational constant, $h > 0$ is the thickness of the fluid layer, $\mathbf{v} = (u,v)$ is the velocity, $\mathbf{b} = (a,b)$ is the magnetic field, and $z(x)$ is the topography.

We introduce a Suliciu type relaxation approximation for the SWMHD system with flat bottom. Under some subcharacteristic conditions, the solver satisfies a discrete entropy inequality, and preserves positivity of density. It resolves exactly all material and Alfven contact discontinuities. Moreover the scheme satisfies an asymptotic consistency with the non-conservative part of the system.

In the case of non-flat bottom, we use the hydrostatic reconstruction method, that leads to a well-balanced scheme with respect to some families of contact discontinuities. Note that the SWMHD system has four linearly degenerate eigenvalues (material, two Alfven, topography waves), that can be resonant. The solver is consistent, satisfies a semi-discrete entropy inequality, and preserves the nonnegativity of the thickness of the fluid layer. In addition, it is well-balanced for resonant material contact discontinuities and resonant material and Alfven contact discontinuities.