An entropy satisfying well-balanced scheme for almost all flow regimes for the Shallow Water equations

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Abstract

This work is devoted to the derivation and analysis of a finite volume method for approximating the solutions of the shallow water equations in one space dimension. By construction, the method is well-balanced for all flow regimes except when resonance takes place. The well-balance property is thus achieved for sub-critical and super-critical solutions at rest and not only for solutions with a null velocity. In addition, the scheme is shown to obey a fully discrete energy like inequality under a classical CFL restriction. These two properties can be met when the exact Riemann solver is used. The latter is known however to be costly and we propose to achieve the same properties with a fairly reduced evaluation cost.

In order to reach the expected simplicity, the self-similar solutions of the exact PDEs are approximated by those of a Suliciu's relaxation system as already suggested by Bouchut [1]. In each relaxation Riemann problem, we propose to model the effect of the varying bathymetry (classically understood in terms of a standing wave) thanks to a Dirac source term concentrated at $x = 0$ which mass has to be suitably prescribed. The additional PDE for governing the relaxation pressure is also equipped with a Dirac source term at $x = 0$ that can be understood as a relaxation defect measure correction. Both masses have to be properly defined depending on the initial data to ensure first some entropy like inequality across the bathymetry standing wave while exactly preserving any given non-resonant exact self-similar solution at rest. This well-balance requirement is known to be equivalent to the property that the entropy law for the exact PDEs (the so-called total energy) is actually a conservation law for such particular solutions. The linear degeneracy of the Suliciu's system, the structure of its relaxation entropy pair and its interplay with the exact entropy pair are seen to provide a valuable tool to achieve both the stability and accuracy requirements in a fairly simple way. Finally, some numerical test cases are presented and strongly support the efficiency of the proposed scheme.

References

[1] F. Bouchut, *Nonlinear Stability of finite Volume Methods for Hyperbolic conservation Laws and well-balanced schemes. Frontiers in Mathematics, Birkhauser (2004).*